

# THE CALABI FLOW ON TORIC FANO SURFACES

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## 1. INTRODUCTION

This is a continuation of the earlier work by the authors on the Calabi flow [9, 10]. We follow the setup of [10], in particular we shall use the result of formation of singularities along the Calabi flow on Kähler surfaces in [10]. The readers are encouraged to consult [10] for the setup and for the references on this topic. The search of extremal Kähler metrics is now a very hot topic in Kähler geometry and many people have been contributing in this effort; we just list only a few references [6, 21, 20, 2, 1, 3, 16] etc.

We believe that the Calabi flow is a very effective tool to approach the existence of extremal metrics on compact Kähler manifolds. One of the main problems on the Calabi flow is the longtime existence. In [9], we proved that the Calabi flow exists as long as Ricci curvature tensors of the evolve metrics stay bounded. This is the first attempt to understand a conjecture by the first named author: starting from any smooth Kähler metric on a compact Kähler manifold (complex dimension of  $n \geq 2$ ), the Calabi flow exists for all positive time. In [10], we focused on the study of the Calabi flow on Kähler surfaces with the assumption that the Sobolev constants of the evolved metrics are uniformly bounded. First we [10] studied the formation of singularity on Kähler surfaces. If the curvature tensor blows up along the Calabi flow, we could then construct a singular model, called a *maximal bubble*, which is a complete asymptotically locally Euclidean (ALE) scalar flat Kähler surface. Then we studied some examples where such a bubble cannot be formed; in particular, we considered a family of Kähler classes on Kähler surfaces of differential type of  $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$  ( $1 \leq k \leq 3$ ). These surfaces are known as del Pezzo surfaces with toric symmetry. We then followed the approach in [11] to analyze all possible maximal bubbles. Actually a maximal bubble can only be formed in a fairly restricted way, in particular with the toric symmetry. With the aid of special geometry of manifolds we considered, in particular the toric symmetry and the discrete symmetry that those Kähler classes admit, we could rule out the formation of maximal bubble. Hence we [10] could prove the longtime existence and convergence of the Calabi flow for those examples. However, the analysis there is quite delicate, complicated and sometime

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outright challenging. It is also very hard to push these ideas beyond the examples we considered in [10], for example for the Kähler classes without discrete symmetry.

In this note we shall adopt a different strategy to rule out possible bubbles; in particular we shall use the toric condition in a more essential way. This allows us to prove some longtime existence and convergence results in a fairly large family of Kähler classes on toric Fano surfaces.

Let  $(M, J)$  be a compact Kähler surface and let  $[\omega]$  be a fixed Kähler class on  $M$ . We shall use  $c_1$  to denote the first Chern class of  $(M, J)$ . We may define a functional

$$\mathcal{B}([\omega]) = 32\pi^2 \left( c_1^2 + \frac{1}{3} \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \right) + \frac{1}{3} \|\mathcal{F}\|^2,$$

where  $\|\mathcal{F}\|^2$  is the norm of Calabi-Futaki invariant [17, 8]. Our main result is

**Theorem 1.1.** *Let  $(M, [\omega], J)$  be a toric Fano surface with positive extremal Hamiltonian potential. If the Calabi flow initiates from a Kähler metric with toric symmetry satisfying*

$$(1.1) \quad \int_M R^2 dg < \mathcal{B}([\omega]),$$

*then the Calabi flow exists for all time and converges subsequently to an extremal metric in  $[\omega]$  in Cheeger-Gromov sense.*

The definition of *extremal Hamiltonian potential* will be given in Section 2. An immediate corollary of Theorem 1.1 is:

**Corollary 1.2.** *Let  $(M, [\omega], J)$  be a toric Fano surface with positive extremal Hamiltonian potential. If there is a toric metric  $\omega_0 \in [\omega]$  such that the Calabi energy of  $\omega_0$  is less than  $\mathcal{B}([\omega])$ , then there exists an extremal metric in  $[\omega]$ .*

**Remark 1.3.** *In [14, 15], Donaldson used a continuity method to deform metrics to seek extremal metrics on toric surfaces and has made striking progress on existence of constant scalar curvature metrics. His approach uses convex analysis, which depends on the fact that, one can express a toric metric on a toric surface in terms of a convex function in a convex polytope in  $\mathbb{R}^2$ .*

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## 2. SOBOLEV CONSTANT

In this section we shall prove that the Sobolev constants of the evolved metrics along the Calabi flow on Fano surfaces are uniformly bounded under certain natural geometric conditions. We shall first define an extremal Hamiltonian potential of an *invariant Kähler metric* in a fixed Kähler class  $(M, [\omega])$ , which is essentially given by [17, 22]. Recall that an extremal vector field for  $(M, [\omega])$  is *a priori* determined [17] up to conjugation. Let  $\mathcal{X}$  be the extremal vector field and let  $\mathcal{X}_{\mathbb{R}}$  be the real part of  $\mathcal{X}$ . Define  $\mathcal{K}_{\mathcal{X}}$  to be the set of all the invariant metrics in  $[\omega]$  which satisfy

$$\mathcal{K}_{\mathcal{X}} = \{\omega : L_{\mathcal{X}_{\mathbb{R}}}\omega = 0\}.$$

For any Kähler metric  $\omega \in \mathcal{K}_{\mathcal{X}}$ , one can define the real potential  $\theta_{\omega}$  [22] by

$$\nabla_{\omega}\theta_{\omega} = \mathcal{X}_{\mathbb{R}}$$

which satisfies the normalized condition

$$\int_M \theta_{\omega} \omega^n = 0.$$

We have the  $L^2$  orthogonal decomposition [19, 8]

$$R_{\omega} = \underline{R} + \theta_{\omega} + \theta_{\omega}^{\perp},$$

where the average of the scalar curvature  $\underline{R}$  is determined by  $(M, [\omega])$ . We can then define the extremal Hamiltonian potential as

**Definition 2.1.** *Let  $\omega \in \mathcal{K}_{\mathcal{X}}$ , the extremal Hamiltonian potential of  $\omega$  is given by*

$$\rho_{\omega} = \underline{R} + \theta_{\omega}.$$

An extremal metric  $\omega$  then satisfies  $R_{\omega} = \rho_{\omega}$ . By its definition [22], the maximum and the minimum of  $\theta_{\omega}$  are the invariants of  $(M, [\omega])$ . We may denote

$$\theta_{-} = \min_{\omega \in \mathcal{K}_{\mathcal{X}}} \theta_{\omega}, \quad \theta_{+} = \max_{\omega \in \mathcal{K}_{\mathcal{X}}} \theta_{\omega}.$$

We can also denote

$$\rho_{-} = \underline{R} + \theta_{-}, \quad \rho_{+} = \underline{R} + \theta_{+}.$$

It is clear that  $\rho_{-}$  and  $\rho_{+}$  are the minimum and maximum of the scalar curvature of an extremal metric respectively if it exists in  $[\omega]$ . If  $(M, [\omega])$  is a Fano surface and the Futaki invariant of  $[\omega]$  is zero, then  $\theta_{-} = \theta_{+} = 0$  and so  $\rho_{-} = \rho_{+} = \underline{R}$  is positive. Hence  $\rho_{-}$  is positive for Fano surfaces of differential type of  $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$  ( $4 \leq k \leq 8$ ). An interesting question is

**Question 2.2.** *Let  $(M, [\omega])$  be a toric Fano surface, is  $\rho_{-}$  positive?*

Note that  $\rho_{-}$  is an invariant of  $(M, [\omega])$  and it is computable in particular when  $(M, J)$  is a toric Fano surface. Hence one can check numerically whether  $\rho_{-}$  is positive or not for any given Kähler class on  $M$ . However, in general it seems not easy to verify

that it is positive since its expression is quite complicated. Without giving detailed argument, S. Simanca claimed that the answer to Question 2.2 to be correct (cf. [24]). Since no detailed computation is given by Simanca in [23, 24], we feel it is important to point out why such a statement is plausible. Note that the average of the scalar curvature on  $(M, [\omega])$  is positive when  $M$  is a Fano surface. Intuitively, if there is an extremal metric, then the scalar curvature of the extremal metric should be positive since it minimizes the Calabi energy. To verify this, one needs to consider the case of  $M \sim \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$  ( $k = 1, 2, 3$ ). When  $k = 1$ , one can check the scalar curvatures of all the extremal metrics constructed by E. Calabi [6] are positive. LeBrun-Simanca [21] computed the Futaki invariant and the extremal vector field of a Kähler class explicitly for Kähler surfaces with a semi-free  $\mathbb{C}^*$  action. In particular their results can be applied to toric Fano surfaces and one can compute further  $\rho_-$ . For example some explicit formula is given in [28]. However, it seems that only when  $(M, [\omega])$  admits some additional discrete symmetry, the formula of  $\rho_-$  is simple enough and one can check directly that it is actually positive. For example, when  $k = 2$ , it is proved that  $\rho_-$  is positive for the bilaterally symmetric Kähler classes [11].

We shall then show how to bound the Sobolev constants on Fano surfaces under natural geometric conditions. The idea dates back to Tian [26] for Kähler metrics of constant scalar curvature (see [27] also) and it is generalized to extremal metrics in Chen-Weber [12].

**Lemma 2.3.** *Let  $(M, [\omega])$  be a Fano surface such that  $\rho_- > 0$  and let  $g$  a Kähler metric in  $[\omega]$ . If  $g$  is invariant ( $g \in \mathcal{K}_{\mathcal{X}}$ ) and*

$$(2.1) \quad \int_M R^2 dg < \mathcal{B}([\omega]),$$

*then the Sobolev constant is bounded a priori as in (2.19).*

When  $g$  is not invariant, Lemma 2.3 still holds with stronger restriction on the Calabi energy. But we shall not need this. We define the Sobolev constant for a compact 4 manifold  $(M, g)$  to be the smallest constant  $C_s$  such that the estimate holds, for any  $f \in W^{2,2}(M, g)$ ,

$$(2.2) \quad \|f\|_{L^4}^2 \leq C_s \left( \|\nabla f\|_{L^2}^2 + V^{-1/2} \|f\|_{L^2}^2 \right),$$

where  $V$  is the volume of the manifold  $(M, g)$ . Note the Sobolev inequality (2.2) is scaling-invariant. When the Yamabe constant is positive, the Sobolev constant is essentially bounded by the Yamabe constant [4]. Recall that the Yamabe constant for a conformal class  $[g]$  of Riemannian metrics on a compact 4 manifold is given by

$$Y_{[g]} = \inf_{\tilde{g} \in [g]} \frac{\int_M R_{\tilde{g}} d\tilde{g}}{\sqrt{\int_M d\tilde{g}}}.$$

By the celebrated work of Trudinger, Aubin and Schoen [5, 25], for any conformal class  $[g]$  the infimum is achieved by the so-called Yamabe minimizer  $g_Y \in [g]$  which necessarily has constant scalar curvature. If  $\tilde{g} = u^2 g$ , the scalar curvature is given by

$$R_{\tilde{g}} = u^{-3}(6\Delta_g u + R_g u),$$

so the Yamabe constant takes the formula

$$(2.3) \quad Y_{[g]} = \inf_{u \neq 0} \frac{\int_M (6|\nabla u|^2 + R_g u^2) dg}{\left(\int_M u^4 dg\right)^{1/2}}.$$

Now we are in the position to prove Lemma 2.3.

*Proof.* We can rewrite (2.1) as

$$(2.4) \quad 96\pi^2 c_1^2 - 2 \int_M R^2 dg > \int_M (R - \underline{R})^2 dg - \|\mathcal{F}\|^2.$$

Following computation in [26, 11] (for example, see Section 5 [11]), we have

$$(2.5) \quad Y_{[g]}^2 \geq 96\pi^2 c_1^2 - 2 \int_M R^2 dg.$$

It then follows from (2.4) and (2.5) that

$$(2.6) \quad Y_{[g]}^2 > \int_M (R - \underline{R})^2 dg - \|\mathcal{F}\|^2.$$

We shall need a decomposition formula of the Calabi energy [19, 8],

$$(2.7) \quad \int_M (R - \underline{R})^2 dg - \|\mathcal{F}\|^2 = \int_M (R - \underline{R} - \theta_\omega)^2 dg.$$

First we show that  $Y_{[g]}$  has to be positive. Pick up a sequence of functions  $u_i$  ( $u_i \neq 0$ ) which minimizes the expression in (2.3). Hence we have

$$(2.8) \quad Y_{[g]} + \epsilon_i = \frac{\int_M (6|\nabla u_i|^2 + R_g u_i^2) dg}{\left(\int_M u_i^4 dg\right)^{1/2}},$$

such that  $\epsilon_i \rightarrow 0$  when  $i \rightarrow \infty$ . We can rewrite (2.8) as

$$(2.9) \quad (Y_{[g]} + \epsilon_i) \|u_i\|_{L^4}^2 = 6 \int_M |\nabla u_i|^2 dg + \int_M R u_i^2 dg,$$

where we write  $R = R_g$  for simplicity. It then follows from (2.9) that

$$(2.10) \quad (Y_{[g]} + \epsilon_i) \|u_i\|_{L^4}^2 - \int_M (R - \underline{R} - \theta_\omega) u_i^2 dg = 6 \|\nabla u_i\|_{L^2}^2 + (\underline{R} + \theta_\omega) \|u_i\|_{L^2}^2.$$

By Cauchy-Schwarz inequality, we compute

$$(2.11) \quad \left| \int_M (R - \underline{R} - \theta_\omega) u_i^2 dg \right| \leq \left( \int_M (R - \underline{R} - \theta_\omega)^2 dg \right)^{1/2} \left( \int_M u_i^4 dg \right)^{1/2}.$$

Then we compute, by (2.11),

$$(2.12) \quad (Y_{[g]} + \epsilon_i) \|u_i\|_{L^4}^2 - \int_M (R - \underline{R} - \theta_\omega) u_i^2 dg \leq (Y_{[g]} + \epsilon_i + \|R - \underline{R} - \theta_\omega\|_{L^2}) \|u_i\|_{L^4}^2.$$

If  $Y_{[g]} < 0$ , then by (2.6) and (2.7), we know that

$$(2.13) \quad Y_{[g]} + \|R - \underline{R} - \theta_\omega\|_{L^2} < 0.$$

Since  $g$  is fixed, then by (2.13),  $Y_{[g]} + \|R - \underline{R} - \theta_\omega\|_{L^2} + \epsilon_i$  is less than zero for sufficiently large  $i$ ; hence by (2.12), we can get that for  $i$  large enough,

$$(2.14) \quad (Y_{[g]} + \epsilon_i) \|u_i\|_{L^4}^2 - \int_M (R - \underline{R} - \theta_\omega) u_i^2 dg < 0.$$

However  $\underline{R} + \theta_\omega \geq \rho_- > 0$ , the right hand side of (2.10) is then positive, which contradicts (2.14). Hence  $Y_{[g]} > 0$ ; it then follows from (2.6) that

$$(2.15) \quad Y_{[g]} > \|R - \underline{R} - \theta_\omega\|_{L^2}.$$

We can then rewrite (2.3) as, for  $u > 0$ ,

$$(2.16) \quad \|u\|_{L^4}^2 \leq \frac{6}{Y_{[g]}} \|\nabla u\|_{L^2}^2 + \frac{1}{Y_{[g]}} \int_M R u^2 dg.$$

It is easy to see that (2.16) holds for any  $u$  since  $|\nabla|u|| \leq |\nabla u|$  at  $u \neq 0$ . Now we rewrite (2.16) as

$$(2.17) \quad \|u\|_{L^4}^2 - \frac{1}{Y_{[g]}} \int_M (R - \underline{R} - \theta_\omega) u^2 dg \leq \frac{6}{Y_{[g]}} \|\nabla u\|_{L^2}^2 + \frac{1}{Y_{[g]}} \int_M (\underline{R} + \theta_\omega) u^2 dg.$$

Note that  $\underline{R} + \theta_\omega \leq \rho_+$ . It follows from (2.17) and Cauchy-Schwarz inequality, that

$$(2.18) \quad \left(1 - \frac{1}{Y_{[g]}} \|R - \underline{R} - \theta_\omega\|_{L^2}\right) \|u\|_{L^4}^2 \leq \frac{6}{Y_{[g]}} \|\nabla u\|_{L^2}^2 + \frac{\rho_+}{Y_{[g]}} \|u\|_{L^2}^2.$$

It then follows from (2.18) that the Sobolev constant of  $g$  is bounded a priori. In other words, we can get that

$$(2.19) \quad C_s \leq \max \left\{ \frac{6}{Y_{[g]} - \|R - \underline{R} - \theta_\omega\|_{L^2}}, \frac{\sqrt{V} \rho_+}{Y_{[g]} - \|R - \underline{R} - \theta_\omega\|_{L^2}} \right\}.$$

□

### 3. RULE OUT BUBBLES

In this section we shall prove Theorem 1.1. First let us recall the formation of singularity along the Calabi flow on Kähler surfaces. Let  $(M, [\omega])$  be a toric Fano surface as in Theorem 1.1. Suppose that the Calabi flow exists on  $[0, T)$ ,  $0 < T \leq \infty$  and the curvature tensor blows up when  $t \rightarrow T$ . Note that under the assumption in Theorem 1.1, the Sobolev constants of the evolved metrics are uniformly bounded by Lemma 2.3, since the Calabi energy is decreasing along the flow. Hence the result

(Theorem 1.1, [10]) is applicable. Since the blowing up process is required in the following argument, we shall state the result as follows.

**Proposition 3.1.** *Keep the assumption in Theorem 1.1. If the curvature blows up when  $t \rightarrow T$ , there exists a sequence of points  $(x_i, t_i) \in (M, [0, T))$  where  $t_i \rightarrow T$  and  $Q_i = \max_{t \leq t_i} |Rm| = |Rm(x_i, t_i)| \rightarrow \infty$  such that the pointed manifolds*

$$(M, x_i, Q_i g(t_i + t/Q_i^2))$$

*converge locally smoothly to an ancient solution of the Calabi flow*

$$(M_\infty, x_\infty, g_\infty(t)), t \in (-\infty, 0].$$

*Moreover,  $g_\infty(t) \equiv g_\infty(0)$  and  $g_\infty := g_\infty(0)$  is a complete scalar flat ALE Kähler metric on  $M_\infty$ .*

One of the key points in [11] is that  $(M_\infty, g_\infty)$ , as a limit of pointed manifolds  $(M, g_i)$ , is toric since  $g_i := Q_i g(t_i)$  is toric. Moreover  $(M_\infty, g_\infty)$  contains holomorphic cycles. The result (Proposition 16, [11]) is only stated for  $M \sim \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ , but the result and the proof hold for all toric Fano surfaces without any change. We shall state the result as follows.

**Proposition 3.2.** *Keep the same assumption as in Theorem 1.1. Suppose the curvature tensor blows up along the Calabi flow and let  $(M_\infty, g_\infty)$  be a maximal bubble. Then  $(M_\infty, g_\infty)$  is toric and  $H_2(M_\infty, \mathbb{Z})$  is generated by holomorphically embedded  $\mathbb{CP}^1$ s in  $M_\infty$ .*

On the other hand, we show that a holomorphic cycle cannot be formed in such a blowup process. The idea is more lucid when the cohomology class  $[\omega]$  is rational.

**Proposition 3.3.** *Keep the same assumption as in Theorem 1.1. Let  $[\omega] \in H^2(M, \mathbb{Q})$ . Then  $(M_\infty, g_\infty)$  cannot contain a holomorphic  $\mathbb{CP}^1$ .*

*Proof.*  $(M_\infty, g_\infty)$  is the limit of pointed manifolds  $(M, g_i)$ . Hence there is a sequence of compact set  $K_i$ ,  $K_i \subset K_{i+1}$ ,  $\cup K_i = M_\infty$ , and a sequence of diffeomorphisms  $\Phi_i : K_i \rightarrow \Phi_i(K_i) \subset M$ ,

$$\Phi_i^*(g_i) \rightarrow g_\infty,$$

where the convergence is smooth in  $K_{i-1}$ . Let  $S$  be an embedded holomorphic  $\mathbb{CP}^1$  in  $M_\infty$ . There is a sequence of compact two spheres, which are denoted as  $S_i = \Phi_i(S)$  and  $S_i \subset \{M, Q_i g(t_i)\}$ . Let  $\omega_\infty$  be the Kähler form of  $g_\infty$  and let  $\omega_i = Q_i \omega(t_i)$  be the Kähler form of  $g_i$ . Since  $\Phi_i^* g_i$  converges to  $g_\infty$  smoothly, then for any fixed positive constant  $\epsilon$  we have

$$(3.1) \quad \left| \int_{S_i} \omega_i - \int_S \omega_\infty \right| = \left| \int_S \Phi_i^* \omega_i - \int_S \omega_\infty \right| < \epsilon$$

when  $i$  is sufficiently large. Hence  $\int_{S_i} \omega_i$  is uniformly bounded and then

$$(3.2) \quad \int_{S_i} \omega(t_i) = \frac{1}{Q_i} \int_{S_i} \omega_i \rightarrow 0.$$

On the other hand, we know that

$$\int_{S_i} \omega(t_i) = \int_{S_i} \omega = [\omega][S_i] = a_i$$

is a constant depending only on  $[\omega], [S_i]$ . Since  $[\omega] \in H^2(M, \mathbb{Q})$ , there exists some  $k \in \mathbb{N}$  such that  $[k\omega] \in H^2(M, \mathbb{Z})$ . It then follows that  $\int_{S_i} k\omega$  is an integer, hence  $ka_i$  is an integer for any  $i$ . By (3.2),  $a_i \rightarrow 0$ , hence  $ka_i$  has to be zero when  $i$  large enough. It then follows that  $a_i = 0$  when  $i$  is sufficiently large. If  $a_i = 0$ , by (3.1), it follows that

$$\int_S \omega_\infty = 0.$$

This contradicts that  $S$  is a holomorphic embedded  $\mathbb{CP}^1$  in  $M_\infty$ .  $\square$

When  $[\omega]$  is not a rational class, the proof is more involved. The key is then to show that  $\{[S_i]\}$  can only contain finite many homology classes, which rely on (3.1), (3.2) and positivity of a Kähler class.

**Proposition 3.4.** *Keep the same assumption as in Theorem 1.1.  $(M_\infty, g_\infty)$  cannot contain a holomorphic  $\mathbb{CP}^1$ .*

*Proof.* Keep the same notations as in Proposition 3.3. It is clear that we can still get (3.1) and (3.2) and when  $i \rightarrow \infty$ ,

$$(3.3) \quad [\omega][S_i] = a_i \rightarrow 0.$$

We show that any such sequence  $\{[S_i]\}$  contains only finite homology classes in  $H_2(M, \mathbb{Z})$ . Recall that the self-intersection of  $S \in H_2(M_\infty, \mathbb{Z})$  is a negative integer [11]. Let  $[S][S] = -k$ , for some fixed integer  $k \geq 1$ . Since the self-intersection is invariant under diffeomorphism, hence for any  $i$ ,

$$(3.4) \quad [S_i][S_i] = -k.$$

The toric Fano surfaces are described as  $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$  ( $\mathbb{CP}^2$  blown up at two distinct points),  $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$  ( $\mathbb{CP}^2$  blown up at three non-linear points). We only exhibit the example when  $M \sim \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ , all other examples are similar (and simpler). Let  $H$  be a hyperplane in  $\mathbb{CP}^2$ .  $M$  can be obtained by blown up at three generic points on  $\mathbb{CP}^2$ . After blown up, we still use  $H$  to denote the corresponding hypersurface on  $M$  and  $E_i, i = 1, 2, 3$  to denote the exceptional divisors. For simplicity, we use  $[H], [E_i]$  to denote the homology classes and their Poincaré dual-the cohomology classes. The Kähler classes on  $M$  can be expressed as

$$[\omega]_{x,y,z} = 3[H] - x[E_1] - y[E_2] - z[E_3].$$



Since  $[\omega]$  is a positive class, then  $x, y, z$  have to satisfy that

$$(3.5) \quad 0 < x, y, z; \text{ and } x + y, y + z, x + z < 3.$$

We can see (3.5) as follows; for example,  $x = [E_1][\omega]_{x,y,z} > 0$  and  $H - E_1 - E_2$  is a holomorphic curve which has area  $3 - x - y$  with respect to  $[\omega]_{x,y,z}$ , hence  $x + y < 3$ . And  $H_2(M, \mathbb{Z})$  can be generated by  $\{[H], [E_i], i = 1, 2, 3\}$ , we can then express  $[S_i]$  as

$$[S_i] = m[H] + n[E_1] + j[E_2] + l[E_3],$$

for some integers  $m, n, j, l$ . We can write (3.3) and (3.4) as, when  $i \rightarrow \infty$ ,

$$(3.6) \quad 3m - nx - jy - lz \rightarrow 0$$

and

$$(3.7) \quad m^2 - n^2 - j^2 - l^2 = -k.$$

We can compute, by (3.6),

$$(3.8) \quad n^2 + j^2 + l^2 \geq \frac{(nx + jy + lz)^2}{x^2 + y^2 + z^2} \rightarrow \frac{9m^2}{x^2 + y^2 + z^2}.$$

Hence, by (3.7) and (3.8),

$$m^2 + k + 1 = n^2 + j^2 + l^2 + 1 \geq \frac{9m^2}{x^2 + y^2 + z^2}.$$

But by (3.5), it is easy to see that

$$x^2 + y^2 + z^2 < 9.$$

For any fixed  $x, y, z$ , it then follows that

$$m^2 \left( \frac{9}{x^2 + y^2 + z^2} - 1 \right) \leq k + 1.$$

It follows that  $m$  has at most finite many solutions. So there are at most finite many  $m, n, j, l$  such that (3.6) and (3.7) are satisfied. It then follows that the homology classes of  $[S_i]$  are finite. Hence we can find a subsequence  $S_{\bar{i}}$  of  $S_i$ , such that  $[S_{\bar{i}}] \in H_2(M, \mathbb{Z})$  has the same homology class for any  $\bar{i}$ . Hence  $a_{\bar{i}} = [\omega][S_{\bar{i}}]$  is a constant independent of  $\bar{i}$ . By (3.3),  $a_{\bar{i}} \equiv 0$ . It then follows that  $[S][\omega_\infty] = 0$  by (3.1). This contradicts that  $S$  is a holomorphic cycle in  $M_\infty$ .  $\square$

**Remark 3.5.** *Similar idea can be applied to the Calabi flow on toric surfaces, if one assumes that the Sobolev constants of the evolved metrics are uniformly bounded.*

Now we shall state a convergence result for the Calabi flow.

**Proposition 3.6.** *Let  $(M, J)$  be a Kähler manifold. Suppose  $(M, g(t), J), 0 \leq t < \infty$  is a solution of the Calabi flow such that the Sobolev constants and the curvature tensors of the evolved metrics are uniformly bounded. Then for every sequence  $t_i \rightarrow$*

$\infty$ , there is a subsequence  $t_{i_k}$  and a sequence of diffeomorphisms  $\Phi_{i_k} : M \rightarrow M$  such that,

$$\Phi_{i_k}^* g(t_{i_k}) \rightarrow g_\infty, \Phi_{i_k}^{-1} \circ J \circ \Phi_{i_k} \rightarrow J_\infty,$$

under a fixed gauge, where the convergence is in  $C^\infty$  topology and  $(M, g_\infty, J_\infty)$  is an extremal Kähler manifold with complex structure  $J_\infty$ .

*Proof.* By assumption both Sobolev constants and curvature tensors are bounded, then all higher derivatives of curvature tensors are uniformly bounded, for example see Lemma 4.2 in [10]. It then follows from the standard ideas in Ricci flow (see Hamilton [18]) to get similar compactness results for the Calabi flow. For a sequence  $t_i \rightarrow \infty$ , there is a subsequence  $t_{i_k} \rightarrow \infty$  such that

$$\{M, g(t + t_{i_k}), -t_{i_k} \leq t \leq 0\} \rightarrow \{M_\infty, g_\infty(t), -\infty \leq t \leq 0\}$$

in Cheeger-Gromov sense. The argument is well known in geometric flows and we shall skip the details. Let  $g_\infty = g_\infty(0)$ ,  $g_{i_k} = g(t_{i_k})$ . In particular,  $(M, g_{i_k}) \rightarrow (M_\infty, g_\infty)$ . Namely, there exists a sequence of diffeomorphisms  $\Phi_{i_k} : M \rightarrow M_\infty$  such that

$$\Phi_{i_k}^* g_{i_k} \rightarrow g_\infty.$$

If necessary, by taking a subsequence, we can get that  $J_{i_k} = \Phi_{i_k}^{-1} \circ J \circ \Phi_{i_k} \rightarrow J_\infty$ . Since  $\nabla_{g_{i_k}} J_{i_k} = 0$ , it follows that  $\nabla_{g_\infty} J_\infty = 0$ , hence  $J_\infty$  is still a complex structure which is compatible with  $g_\infty$ . We then show  $g_\infty$  is an extremal metric. This follows from that the Calabi flow is the gradient flow of the Calabi energy. For any  $t_0 \in (-\infty, 0]$ , we choose the sequence  $\{t_{i_k}\}$  such that  $t_{i_k} < t_{i_{k+1}} + t_0$ . Let  $\mathcal{C}(g)$  be the Calabi energy of  $g$ . Since the Calabi energy is decreasing along the Calabi flow, we have

$$\mathcal{C}(g_\infty) = \lim_{t_{i_k} \rightarrow \infty} \mathcal{C}(g(t_{i_k})) \geq \lim_{t_{i_{k+1}} \rightarrow \infty} \mathcal{C}(g(t_0 + t_{i_{k+1}})) = \mathcal{C}(g_\infty(t_0)).$$

It then follows that  $g_\infty(t)$  is an extremal metric for any  $t \in (-\infty, 0]$ .  $\square$

**Remark 3.7.** In general  $J_\infty$  does not have to be the same as  $J$ .

Now we are in the position to prove Theorem 1.1. We argue by contradiction.

*Proof.* By Lemma 2.3, the Sobolev constants of evolved metrics are uniformly bounded under the assumption in Theorem 1.1. If the curvature tensors are not uniformly bounded, there is a contradiction by Proposition 3.1, 3.2 and 3.4. Hence the curvature tensors have to be uniformly bounded and the Calabi flow exists for all time. It then follows that  $(M, g(t), J)$  converges to an extremal metric  $(M, g_\infty, J_\infty)$  subsequently in Cheeger-Gromov sense by Proposition 3.6. We then finish the proof by showing that  $(M, J_\infty)$  is biholomorphic to  $(M, J)$ . The proof follows from [11] (Theorem 27) by using the toric condition carefully and the classification of complex surface. Theorem 27 in [11] states only for  $M \sim \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$  but the proof holds for all toric Fano surfaces. The key is that in the limiting process, the torus action converges and  $(M, g_\infty, J_\infty)$  is still toric. Moreover, the 2-torus action for  $(M, g_\infty, J_\infty)$  is holomorphic with respect

to  $J_\infty$ . We shall sketch the argument for  $M \sim \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ . The readers can refer to [11] for details. When  $M \sim \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ , each of holomorphic curves  $H, E_1, E_2, E_3$  is the fixed point set of the isometric action of some circle action of 2-torus, and so each is totally geodesic with respect to the metrics along the Calabi flow. By looking at the corresponding fixed points set of the limit action of circle subgroups, we can find corresponding totally geodesic 2-spheres in  $(M, g_\infty, J_\infty)$  which are the limits of the image of these submanifolds. Moreover, these limit 2-spheres are holomorphic with respect to  $J_\infty$  and the homological intersection numbers of these holomorphic spheres do not vary. Namely, we have still three holomorphic  $\mathbb{CP}^1$ s with self-intersection  $-1$  as the images of the original exceptional divisors  $E_1, E_2, E_3$ . Thus, by blowing down the images of  $E_1, E_2, E_3$  and applying the classification of the complex surface, we conclude that  $(M, J_\infty)$  is biholomorphic to  $\mathbb{CP}^2$  blown up three generic points. So there exists a diffeomorphism  $\Psi$  such that  $\Psi_* J = J_\infty$ . So  $\Psi^* g_\infty$  is an extremal metric in the class  $[\omega]$  for  $(M, J)$ .  $\square$

**Remark 3.8.** *We may define a functional*

$$\mathcal{A}[\omega] = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2} \|\mathcal{F}\|^2.$$

*This functional has the important property [8, 13] that any Kähler metric  $g$  in the class  $[\omega]$  satisfies the curvature inequality*

$$\int_M R^2 dg \geq 32\pi^2 \mathcal{A}([\omega])$$

*with equality if and only if  $g$  is an extremal metric. A necessary condition for (1.1) to hold is that  $(M, [\omega])$  satisfies the generalized Tian's condition in [12],*

$$c_1^2 > \frac{2}{3} \mathcal{A}([\omega]).$$

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